Exercise 2

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - 3y' = \sin 2x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 3y_c' = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - 3(re^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 3r = 0$$

Solve for r.

$$r(r-3) = 0$$

$$r = \{0, 3\}$$

Two solutions to the ODE are $e^0 = 1$ and e^{3x}; by the principle of superposition, then,

$$y_c(x) = C_1 + C_2 e^{3x}.$$

The particular solution satisfies the original ODE.

$$y_p'' - 3y_p' = \sin 2x \tag{2}$$

Since the inhomogeneous term is a sine and there are even and odd derivatives, the particular solution is $y_p = A\cos 2x + B\sin 2x$.

$$y_p = A\cos 2x + B\sin 2x \quad \rightarrow \quad y_p' = -2A\sin 2x + 2B\cos 2x \quad \rightarrow \quad y_p'' = -4A\cos 2x - 4B\sin 2x$$

Substitute these formulas into equation (2).

$$(-4A\cos 2x - 4B\sin 2x) - 3(-2A\sin 2x + 2B\cos 2x) = \sin 2x$$

$$(-4A - 6B)\cos 2x + (-4B + 6A)\sin 2x = \sin 2x$$

Match the coefficients on both sides to get a system of equations for A and B.

$$\left.\begin{aligned} -4A - 6B &= 0 \\ -4B + 6A &= 1 \end{aligned}\right\}$$

Solving this system yields

$$A = \frac{3}{26} \quad \text{and} \quad B = -\frac{1}{13},$$

which means the particular solution is

$$y_p = \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 + C_2 e^{3x} + \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.